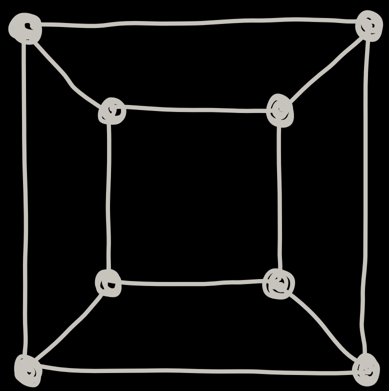
24/10/22

3. 3-polytopes & edge-graphs

3.1 Characterizing 3-polytopes



edge-graph of 3-cube

Def: given a polytope $P \subset \mathbb{R}^d$, its edge-graph $G_P = (V, E)$ has vertex set $V := \mathcal{F}_0(P)$ and $v, w \in V$ adjacent iff $\operatorname{conv}\{v, w\} \in \mathcal{F}_1(P)$

edges of P

→ this graph looks almost like we could read the polytope from it ... can we?

MAIN QUESTIONS

- given a graph, is it the edge-graph of a (3-) polytope?
- given an edge-graph, can I reconstruct the face lattice from it?

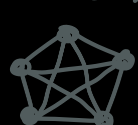
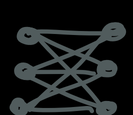
→ Thes questions are essentially answered for 3-polytopes

Thm: (Steinitz)

G is the edge-graph of a 3-polytope iff G is 3-connected + planar.

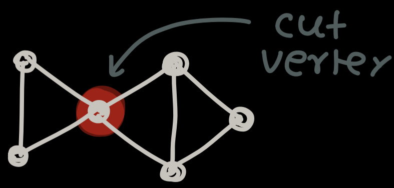
can be drawn in the plane without intersecting edges

non-planar

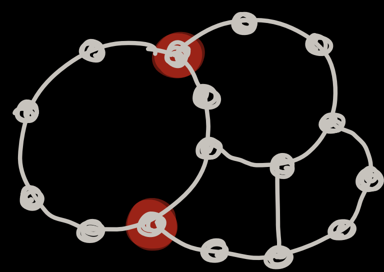
K_5   $K_{3,3}$

Def: a graph on at least $k+1$ vertices is **k-connected** if deletion of any $k-1$ vertices yields a connected graph

Ex: • 1-connected = connected



1- connected but not
2- connected

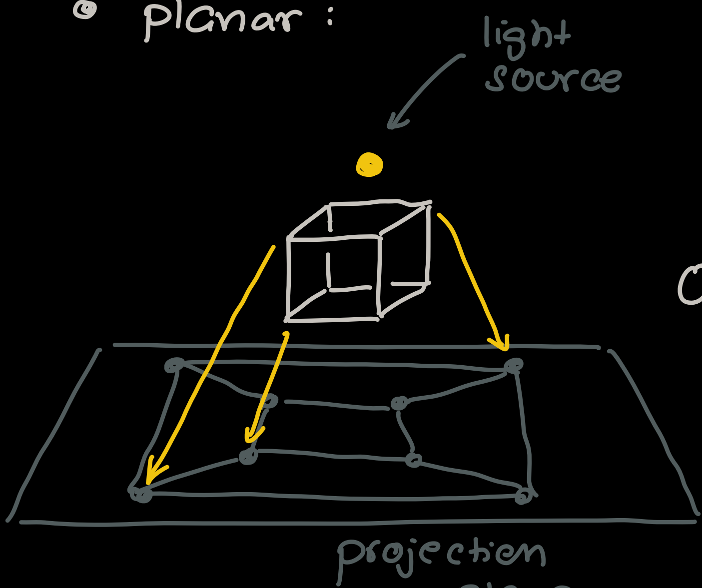


2-connected but not
3-connected

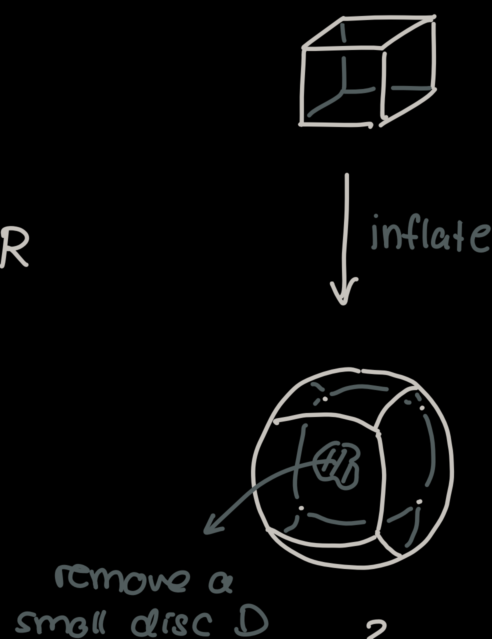
Proof: (of Steinitz)

$\Longrightarrow$: start from a 3-polytope $P \subset \mathbb{R}^3$. Then G_P is

- 3-connected: see section 3.2.
- planar:



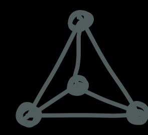
OR

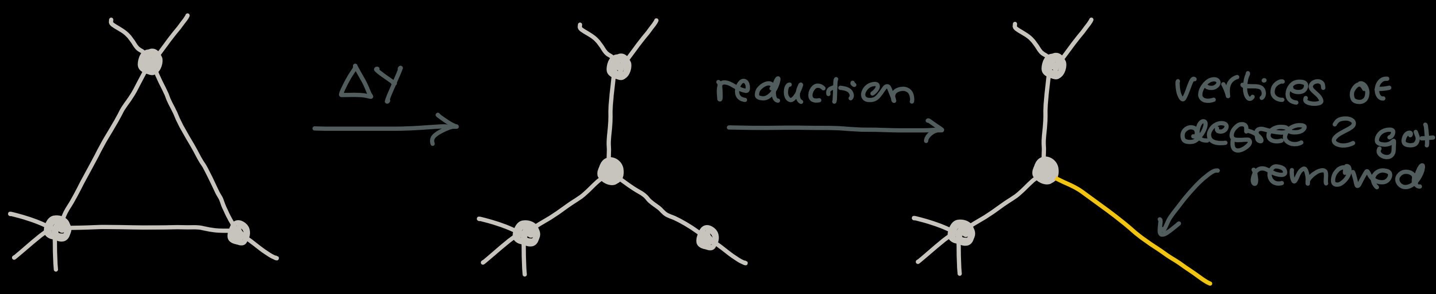


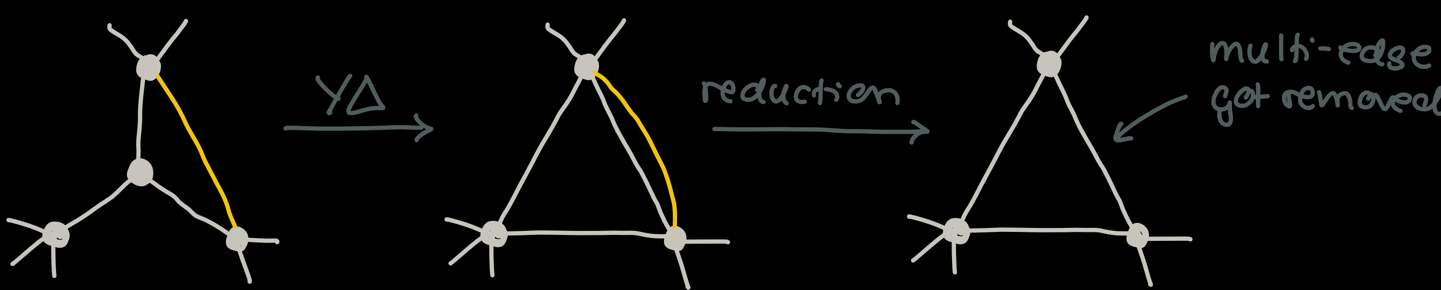
$S^2 \setminus D \simeq \mathbb{R}^2$

$\Longleftarrow$: we use a structure theorem for 3-connected planar graphs (see Ziegler section 4.3)

Thm. Every 3-connected planar graph can be transformed into K_4 using reduced ΔY- and $Y\Delta$-transforms.

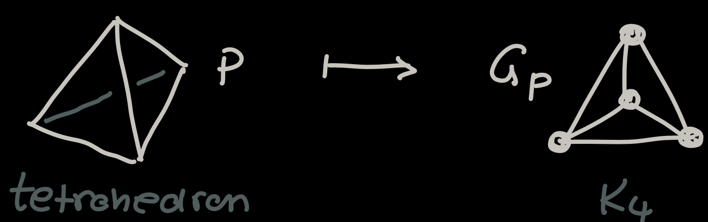
complete graph on 4 vertices 





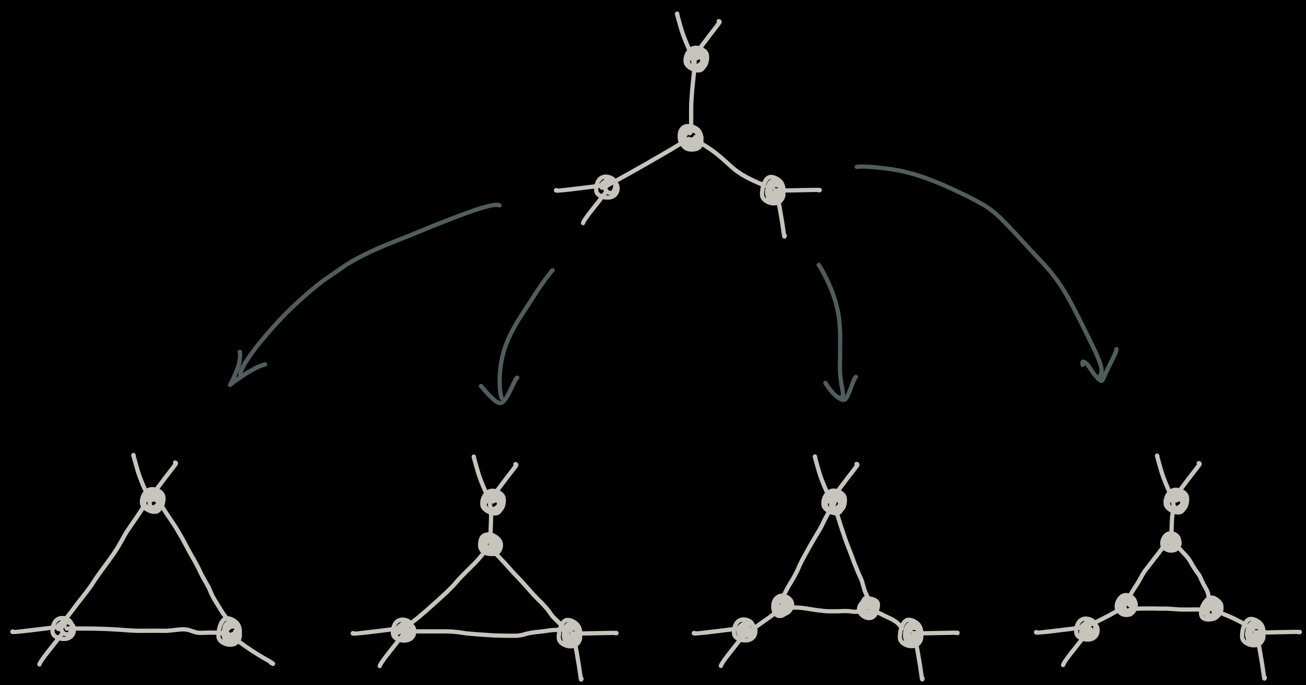
Idea:

- every 3-connected planar graph can be obtained from K_4 using "inverted reduced ΔY- and $Y\Delta$-trafos".
- K_4 is an edge-graph of a 3-polytope:



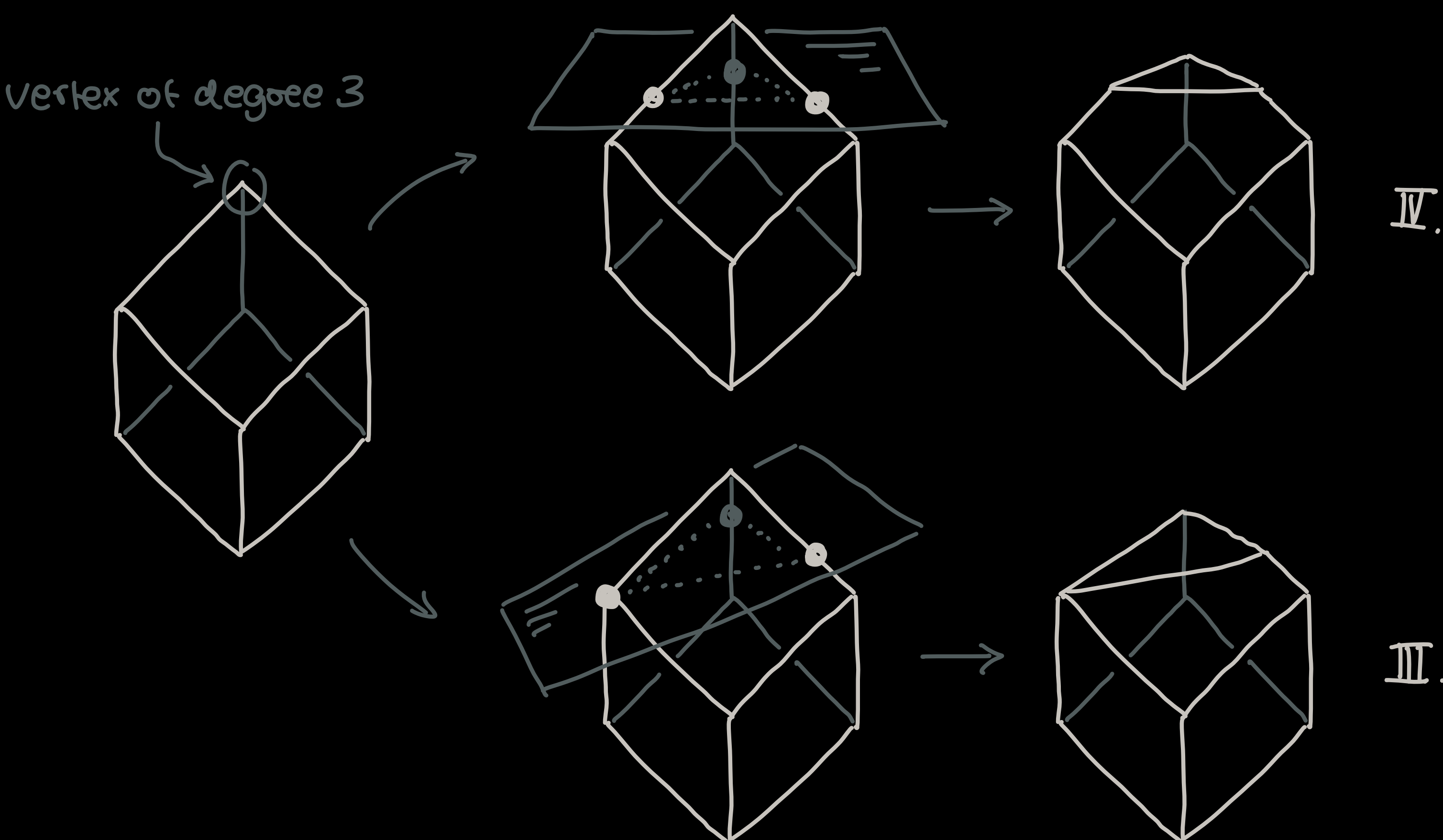
- can we perform the "inverted reduced trafos" on the polytopes as well? Yes (see below)
- BUT: an "inverted reduced trafo" can have more than one result!

Inverse of ΔY :

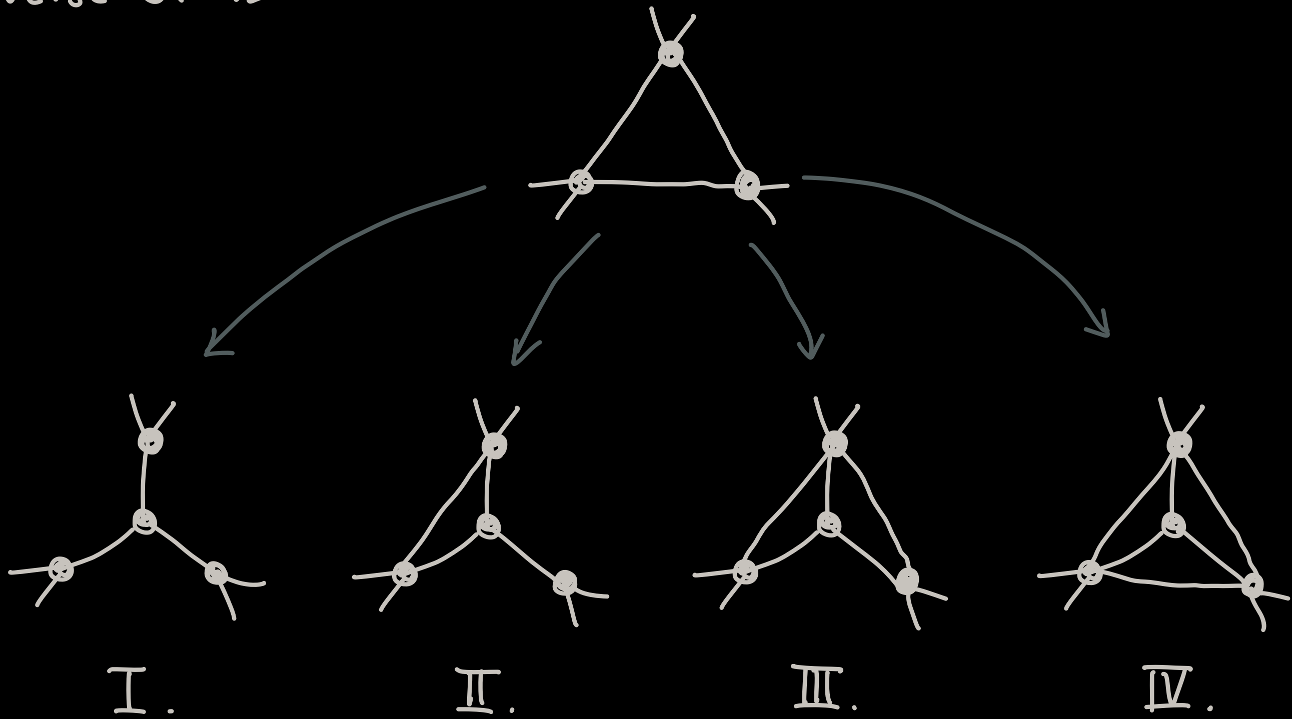


I. II. III. IV.

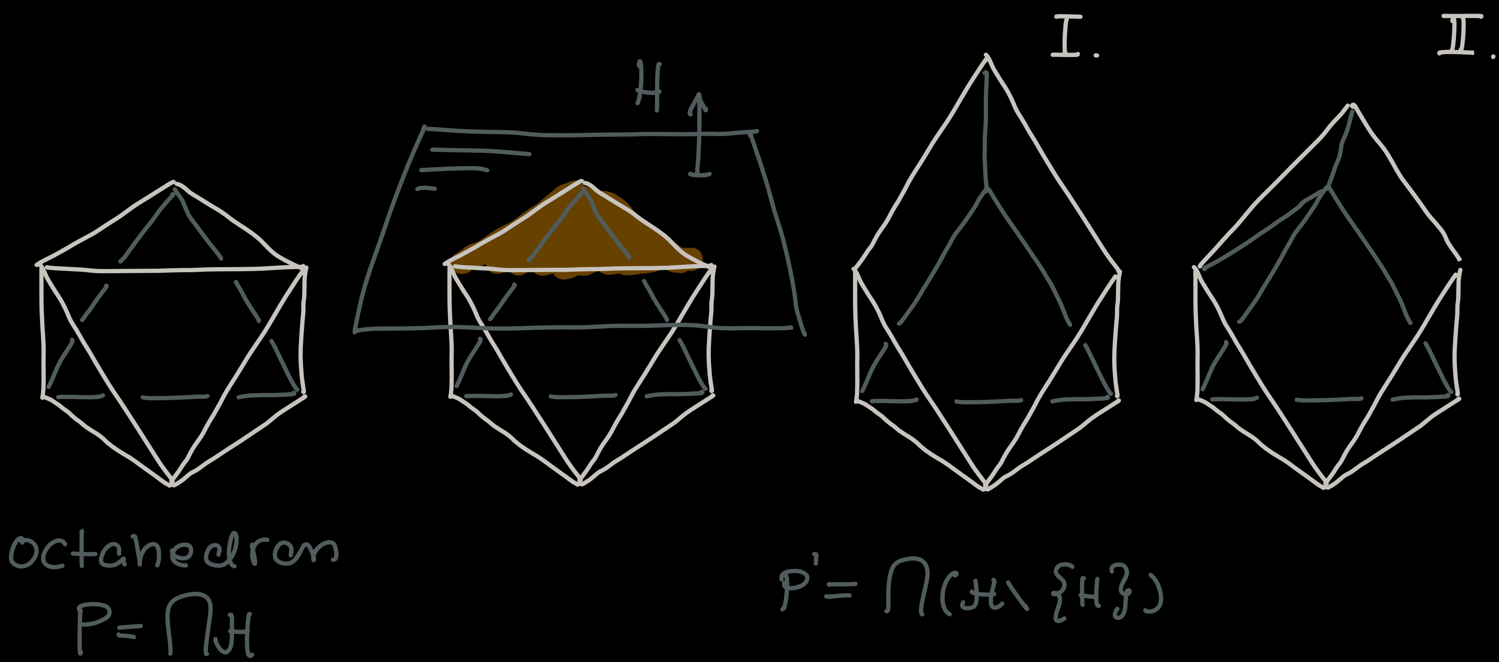
$\rightarrow$ cut off a vertex of degree 3



Inverse of $Y\Delta$



→ remove a facet-defining halfspace from $\mathcal{H}$-representation



$\Rightarrow$ every 3-connected planar graph is the edge-graph of a 3-polytope! $\square$

Extensions: there always exists a 3-polytope ...

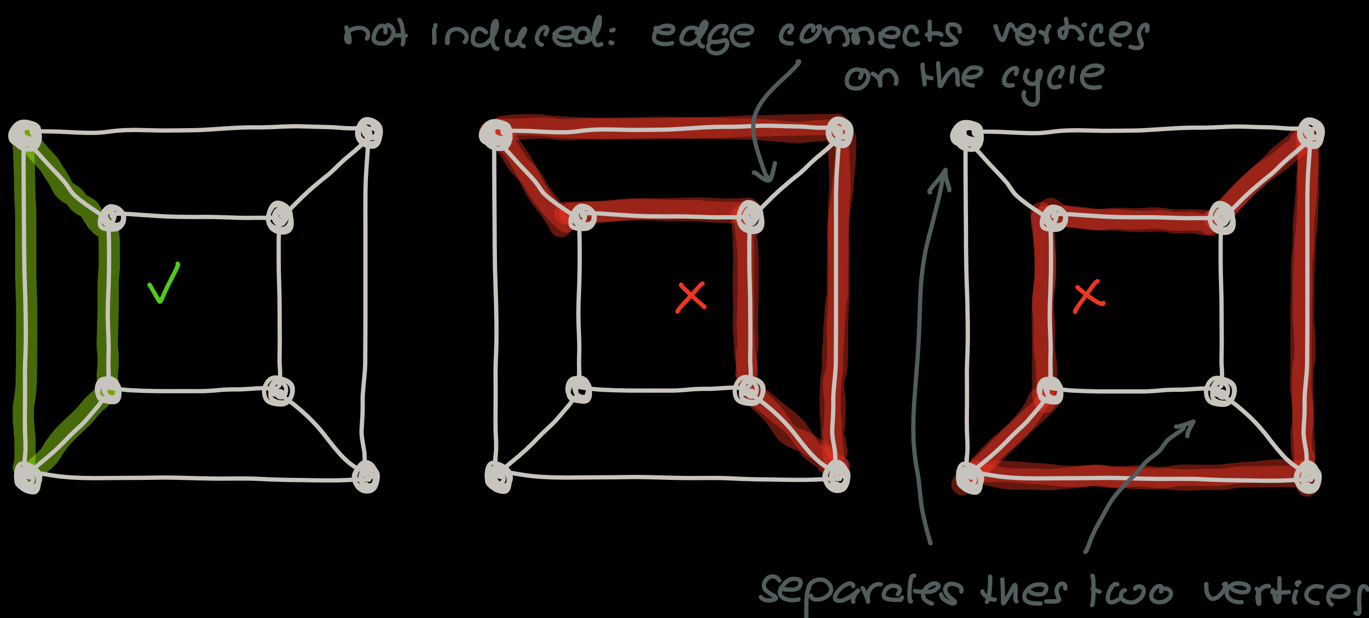
- with all vertices and normal vectors beeing rational
- being maximally symmetric
  = as symmetric as the edge-graph
- one face of which can be arbitrarily prescribed
- which has an edge in-sphere
  := a sphere that touches each edge
  (canonical polyhedron)

combinatorial

There is a quick way to tell the faces from the edge-graph.

Thm: (Tutte)

A cycle $C \subseteq G_P$ corresponds to a face of P iff C is non-separating and induced.



3.2 Edge-graphs of higher-dimensional polytopes

- The edge-graph has worked so well as a tool to understand 3-polytopes.
- Does it work as well for higher-dimensional polytopes?

  → as far as we know NO :(

  → edge-graph seems to tell little in dim ≥ 4

  → little is known about the structure of edge-graphs in dim ≥ 4

- we discuss some of the few things that are known

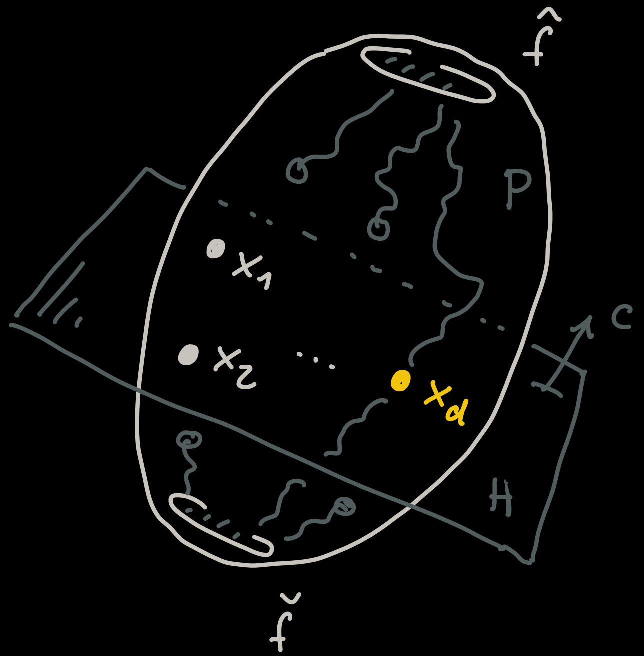
Thm: (Balinski)
The edge-graph of a d-polytope is d-connected

Proof:

- fix d-polytope $P \subset \mathbb{R}^d$ and $d-1$ vertices $x_1, \dots, x_{d-1} \in \mathcal{F}_0(P)$

  → we need to show: $G_P - \{x_1, \dots, x_{d-1}\}$ is connected

- fix arbitrary other vertex $x_d \in \mathcal{F}_0(P)$
- there exists a unique hyperplane H through $x_1, \dots, x_d$ with normal vector $c \in \mathbb{R}^d$
- let $\hat{f}$ (resp. $\check{f}$) be the top (resp. bottom) face of P w.r.t. the direction c



- suppose $\hat{f}, \check{f} \not\subset H$
- we show $(*)$:

  every vertex above (or in) H has a path to $\hat{f}$

- likewise below H
- also: $\hat{f}, \check{f}$ have connected edge-graphs (by induction)

$\to$ connectivity of $G_P - \{x_1, \dots, x_{d-1}\}$ follows

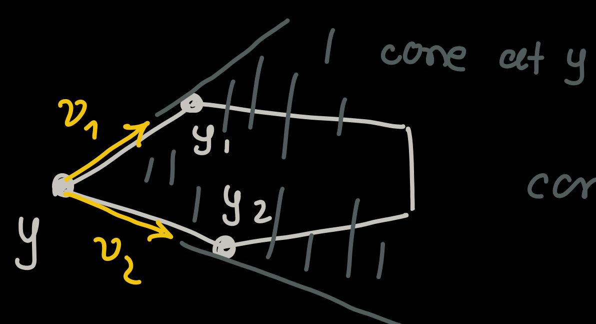
- it remains to show $(*)$

Ex: what if $\hat{f} \subset H$?

- w.l.o.g. we show that every vertex $y \in \mathcal{F}_0(P)$ above (or in) H has an edge "going upwards"
- we use vertex cones:

Given a vertex $y \in \mathcal{F}_0(P)$, let $y_1, \dots, y_r \in \mathcal{F}_0(P)$ be its neighbors in G_P. Let $v_i := y_i - y$ be the directions of edges emanating from y.

Then: $P \subset y + \operatorname{cone}\{v_1, \dots, v_r\}$ Ex: prove this.



$\operatorname{cone}\{v_1, \dots, v_r\} := \{ \sum \alpha_i v_i \mid \alpha_i \geq 0 \}$

... cone spanned by the v_i

- if no edge from y is pointing "upwards" ...

... then the cone at y contains no points "above y". Since P is in the cone, y must already be at the top. $\square$

(all of this is pretty straight forward if you know about the simplex algorithm)

<u>Consequences</u>:

- edge-graphs are connected
- minimum <u>degree</u> of edge-graph of d-poly. is $\geq d$

  # edges incident to vertex

$\longrightarrow$ polytopes with the minimal degree everywhere have special significance

<u>Def</u>: A d-polytope is simple if the edge-graph is d-regular := every vertex degree is d.

<u>E.g.</u> d-cube and d-simplex are simple

d-crosspolytope only for $d \leq 2$

<u>Ex</u>: faces of simple polytopes are simple.

- There exists a dual notion to simplicity

Def: A polytope is simplicial if every facet is a simplex. Ex: every face is a simplex

E.g. d-crosspolytope and d-simplex are simplicial
d-cube only for $d \leq 2$

Ex: only polytope which is both simple and simplicial is simplex

Ex: polar duals of simple polytopes are simplicial and vice versa.

- simple / simplicial polytopes are important because they are generic:

  - choose some random points in $\mathbb{R}^d$; their convex hull is simplicial with probability 1 (because prob. that more than d points lie in a facet defining hyperplane = 0)
  - choose some random halfspaces of $\mathbb{R}^d$; their intersection (if bounded) is simple w.p. 1 (because prob. that more than d hyperplanes intersect at a common point = 0)

We will see:

- edge-graph contains a lot information for simple polytopes, but almost none for simplicial

3.3 Neighborly and cyclic polytopes

- simplices have a very special edge-graph: any two vertices are adjacent $\longrightarrow$ K_n complete graph on n vertices

Q: Can there be other polytopes with complete edge-graph?

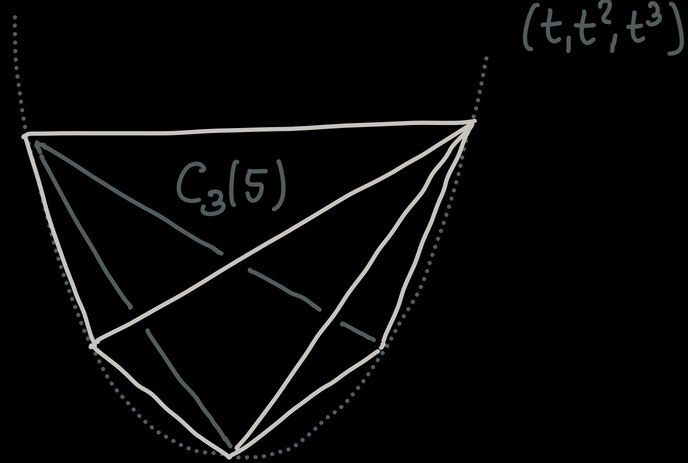
- NO in dimension 3 Ex: show using $V-E+F=2$
- surprisingly YES in dimension ≥ 4; in fact, a "random combinatorial type" has complete edge-graph with probability $\to 1$.

Def: A polytope is k-neighborly if any $\leq k$ vertices form a face.

- 1-neighborly means nothing (every vertex is a face)
- 2-neighborly = edge-graph is complete
  = often just "neighborly"
- We discuss the most famous class of neighborly polytopes

Def:
- the moment curve is the curve $x: \mathbb{R} \to \mathbb{R}^d$ with
$$x(t) := (t, t^2, \ldots, t^d)$$
- for $t_1 < t_2 < \cdots < t_n$, the cyclic polytope of dimension d with n vertices $(n \geq d+1)$ is
$$C_d(n) := \operatorname{conv}\{x(t_i) \mid i \in [n]\}$$

$\longrightarrow$ We shall see: combinatorial type is independent of the choice of the t_i



Lem: Cyclic polytopes are simplicial.

Proof:

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ x(s_0) & & x(s_d) \end{pmatrix} = \det \begin{array}{c} \overbrace{\begin{pmatrix} 1 & & 1 \\ s_0 & & s_d \\ s_0^2 & \cdots & s_d^2 \\ \vdots & & \vdots \\ s_0^d & & s_d^d \end{pmatrix}}^{d+1} \end{array} \left.\vphantom{\begin{pmatrix} 1 \\ s_0 \\ s_0^2 \\ \vdots \\ s_0^d \end{pmatrix}}\right\} d+1$$

Vandermonde identity

$$= \prod_{0 \le i < j \le d} (s_i - s_j)$$

Ex: try to prove this
(both sides are polynomials that vanish if $s_i = s_j$ for some $i<j$)

$\neq 0$ if all s_i are distinct

$\Longrightarrow$ no $d+1$ distinct points on the moment curve are on the same hyperplane

$\Longrightarrow$ a facet can contain $\underbrace{\text{at most } d \text{ points}}_{(d-1)-\text{simplex}}$ $\square$

Thm: (Gale's evenness criterion)

$\longrightarrow$ an algorithm to find out which subsets $S \subset [n]$ with $|S| = d$ form a facet of $C_d(n)$.

1) write S as a characteristic vector

$$\chi_S = (1\,0\,0\,1\,1\,0\,0\,0\,1\,0\,0\,1\,1\,1\,1)$$

$$= \Big(\underset{1}{\bullet} \quad 2 \quad 3 \quad \underbrace{\underset{4}{\bullet} \quad \underset{5}{\bullet}}_{\text{even}} \quad 6 \quad 7 \quad 8 \quad \underbrace{\underset{9}{\bullet}}_{\text{odd}} \quad 10 \quad 11 \quad \underbrace{\underset{12}{\bullet} \quad \underset{13}{\bullet} \quad \underset{14}{\bullet} \quad \underset{15}{\bullet}}_{\text{not inner}} \Big)$$

2) $F := \operatorname{conv}\{x(t_i) \mid i \in S\}$ is a facet of $C_d(n)$ iff all inner blocks of S are of even size.

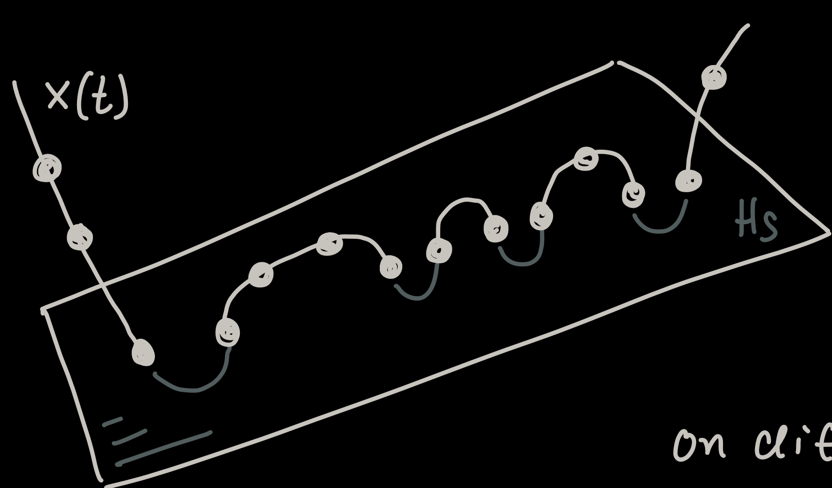
Proof: $S = \{i_1, \dots, i_d\}$

- let H_S be the unique hyperplane through the $x(t_i)$, $i \in S$.
- we can write

some non-zero linear functional

$$H_S = \{x \in \mathbb{R}^d \mid F_S(x) = 0\}$$

with $F_S(x) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x & x(t_{i_1}) & \cdots & x(t_{i_d}) \end{pmatrix}$ NOTE: linear in x



- If we always hit H_S in an even number of consecutive vertices, no two vertices can be on different sides of H_S. $\square$

→ knowing all facets it is not hard to derive all other faces

Thm: $C_d(n)$ is $\lfloor d/2 \rfloor$-neighborly.

Proof idea: (not contained in the lecture)

- choose any subset $S \subset [n]$ of size $\lfloor d/2 \rfloor$
- show that one can always embed S in a larger set $\bar{S}$ of size d with no odd inner blocks.

→ $\bar{S}$ is a facet, hence a simplex

→ S is a face of this simplex, hence a face of $C_d(n)$.

□

Remarks:

- this is as neighborly as a polytope can become without being a simplex:

$$P \text{ is } > \lfloor d/2 \rfloor\text{-neighborly} \longrightarrow P = \text{simplex}$$

(proof: next week)

- $C_d(n)$ cannot be distinguished from a simplex by its $\lfloor d/2 \rfloor$-skeleton := poset of faces up to dimension $\lfloor d/2 \rfloor$
- if $d=3$ then $\lfloor d/2 \rfloor = 1$

→ 1-neighborly means nothing

3.4. Reconstruction from edge-graphs and skeleta

- We have seen that a general polytope (i.e. its combinatorial type) cannot be reconstructed from the edge-graph or even the $\lfloor d/2 \rfloor$-skeleton
- not even the dimension can be reconstructed !!

$C_4(7)$ and 6-simplex have same graph
4-dimensional 6-dimensional K_7

OTHER RESULTS:

- reconstruction is always possible from $(d-1)$- or $(d-2)$-skeleton.
(classic) (by Margaret Bayer)
- reconstruction not possible from $(d-3)$-skeleton.

Thm: (Blind & Mani; proof by Kalai)
"Kalai's simple way to tell a simple polytope from its graph"

If $P, Q \subset \mathbb{R}^d$ are simple with the same edge-graph, then P, Q are combinatorially equiv.

Proof idea: (not contained in the lecture)

we find a combinatorial criterion for when a subset of vertices forms a face (cf. Tutte crit. for 3-poly.)

- consider acyclic orientations of the edge-graph.
  no directed cycles
- for orientation O set

$$h_i^O := \#\text{vertices with out-degree } i .$$

- an acyclic orientation is good if it minimizes

$$h_0^O + 2h_1^O + 4h_2^O + \cdots + 2^d h_d^O .$$

- a connected regular subgraph $H \subseteq G_P$ belongs to a face iff it is terminal w.r.t. some good orientation   no edge leading out of it. $\square$

REMARKS:

- reconstruction also possible with up to 2 non-simple vertices
- but not with 3 non-simple vertices

Joseph Doolittle

- Kalai's proof computationally inefficient but better algorithms exist.
- Known techniques cannot be used to tell whether a regular graph belongs to a simple polytope

OPEN: Can this question be decided efficiently?